Classical Analogy of Fano Interference

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Abstract

An analogy of the quantum mechanical Fano interference to the classical system of harmonic oscillators is presented. I take the coherent atom-laser interaction to illustrate the Fano interference in quantum mechanics and then the analogy between the dressed state picture of coherent-atom laser interaction to the classical coupled harmonic oscillators is described. The Autler-Townes splitting in the coherent atom-laser system is analogous to the normal-mode splitting in the oscillators system though the reasons are completely different. I experimentally demonstrate this phenomena using the coupled electrical resonator circuit.

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1 Introduction

In Physics, Interference is a well known phenomena where interaction of correlated or coherent waves gives rise to the intensity variation of the resultant wave in the form of a symmetric line profiles. However, in certain cases asymmetric line profiles are obtained as a result of the interference among waves, which is known as Fano Interference, named after Italian physicist Ugo Fano. This phenomena is generally regarded as a purely quantum mechanical phenomena being caused by the interference of the waves coming from different channels(unlike the Fabry-Perot interferometre in optics where resonance occurs due to interference of two counter waves in the same scattering channel). From the quantum mechanical point of view, in the Fano resonance, two paths,one direct from the discrete state and the other mediated by the continuum - interfere to produce the asymmetric Fano profile given by :

$$f(\epsilon) = \frac{(\epsilon + q)^2}{1 + \epsilon^2} \tag{1}$$

where, the dimensionless energy parameter $\epsilon = \frac{E - E_R}{\Gamma}$ is used to measure the energy difference between the energy E and the energy of the peak position E_R, and Γ is the width of the resonance. q is the asymmetry parameter measuring quantitatively the degree of asymmetry of the resonance profile. If the asymmetry parameter becomes either very large($|q| \to \infty$) or very small($|q| \to 0$), then the Fano profile reduces to the symmetric Breit-Wigner or Lorentzian profile.

This phenomena has been observed in many quantum mechanical systems, like semiconductor quantum wells, gold nanoparticles, nanomechanical systems etc. Electromagnetically induced transparency (EIT) is also a manifestation of the Fano interference.

The simple classical theory of the driven damped coupled oscillator tells us that the line profile of the oscillator system is also asymmetric, very much analogous to the quantum mechanical Fano asymmetric profiles! And in fact, we can identify the multiple energy-transfer pathways(between the source and the oscillator) which upon interference give rise to the asymmetric profile.

We begin our description with the simple analysis of the classical driven damped coupled oscillator system to understand the classical reason behind this asymmetric profile. Then we will examine the Fano Interference in quantum mechanical system. After that, we will discuss the analogy between the classical oscillator system and the quantum mechanical coherent atom - laser interaction. At last, we will discuss the Fano interference in a coupled resonator circuit through experimental demonstration.

2 Resonance in the Driven Damped Coupled Oscillator System

2.1 A Single Oscillator

The dynamics of a single harmonic oscillator under the presence of the damping & sinusoidal driving force is governed by the differential equation:

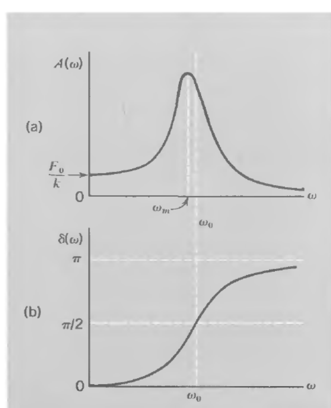
$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = A_0 \cos\omega t = \frac{F_0}{m}\cos\omega t \tag{2}$$

where, ω_0 is the natural frequency of the oscillator in the absence of the damping & driving force, γ is the proportionality constant of the damping force, and ω is the frequency of the external periodic force.

The steady state solution is given by:

$$x(t) = |A(\omega)|\cos[\omega t - \delta(\omega)] \tag{3}$$

where, $|A(\omega)| = \frac{A_0/2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$,$\delta(\omega) = \tan^{-1}(\frac{\omega\gamma}{\omega_0^2-\omega^2})$



So, when the external driving frequency becomes **almost** equal to the natural frequency($\omega \approx \omega_0$), then resonance occurs.Also, from the figure it is clear that the phase of the oscillator changes by π when the oscillator goes through the resonance(remember δ is the angle by which the driving force leads the displacement). As ω increases, the phase increases from 0 at $\omega = 0$ to $\pi/2$ at $\omega = \omega_0$ and to π as $\omega \to \infty$. This is a clear indication of the delay between the action of the driving force & the response of the oscillator with the increase of the external frequency.

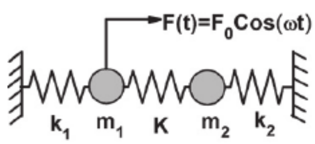
2.2 Two Coupled Oscillators

The dynamics of two coupled harmonic oscillators under the presence of the damping & sinusoidal driving force(to one of them) is governed by the differential equations:

$$\ddot{x_1} + \gamma_1 \dot{x_1} + \omega_{10}^2 x_1 + v_{12} x_2 = A_1 \cos\omega t = \frac{F_0}{m_1}\cos\omega t \tag{4}$$

$$\ddot{x_2} + \gamma_2 \dot{x_2} + \omega_{20}^2 x_2 + v_{12} x_1 = 0 \tag{5}$$

where, v_{12} describes the coupling of the two oscillators.



In the absence of the coupling ($v_{12} = 0$), the two oscillators oscillate independently with their natural frequencies (ω_{10} & ω_{20}). But, in the presence of the coupling the system has two normal modes : two oscillators swing back and forth together (in-phase mode) & they move in opposite directions (out-phase mode).

In order to understand the behaviour of the system at the normal modes let's first consider the case when there is **no damping** i.e. $\gamma_1 = 0 = \gamma_2$. Then the eigen-frequencies can be obtained from :

$$(\omega_{10}^2 - \omega^2)(\omega_{20}^2 - \omega^2) - v_{12}^2 = 0 \tag{6}$$

If the coupling parameter is weak ($\omega_2^2 - \omega_1^2 >> v_{12}$), then the eigen-frequencies are given by:

$$\tilde{\omega}_1^2 \approx \omega_1^2 - \frac{v_{12}^2}{\omega_2^2 - \omega_1^2} \tag{7}$$

$$\tilde{\omega}_2^2 \approx \omega_2^2 + \frac{v_{12}^2}{\omega_2^2 - \omega_1^2} \tag{8}$$

which are slightly shifted from the free mode frequencies of the individual oscillators.

Now, we consider the general case where the damping is non-zero. The steady state solutions are obtained by putting $x_1(t) = c_1 \cos\omega t$ & $x_2(t) = c_2 \cos\omega t$ in (4) & (5). Upon solving those we get :

$$c_1 = \frac{(\omega_2^2 - \omega^2 + i\gamma_2\omega)}{(\omega_1^2 - \omega^2 + i\gamma_1\omega)(\omega_2^2 - \omega^2 + i\gamma_2\omega) - v_{12}^2} A_1 \tag{9}$$

$$c_2 = -\frac{v_{12}}{(\omega_1^2 - \omega^2 + i\gamma_1\omega)(\omega_2^2 - \omega^2 + i\gamma_2\omega) - v_{12}^2} A_1 \tag{10}$$

The phase of the amplitude of the oscillators are defined through $c_1(\omega) = |c_1(\omega)|e^{-i\phi_1(\omega)}$ & $c_2(\omega) = |c_2(\omega)|e^{-i\phi_2(\omega)}$. The phase difference between the two oscillators is given by

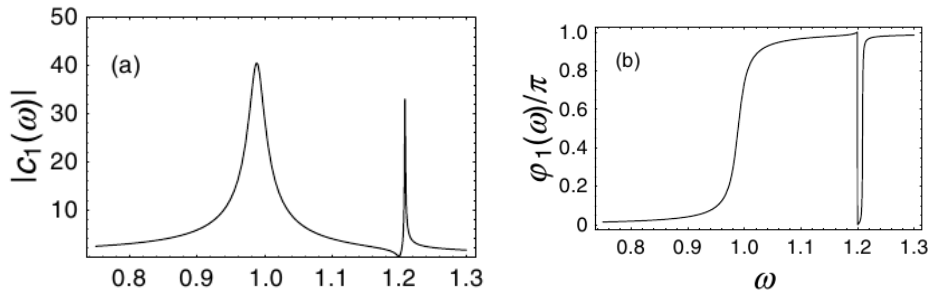
$$\phi_2 - \phi_1 = \pi - \theta \tag{11}$$

where,

$$\theta = \tan^{-1}\left(\frac{\gamma_2\omega}{\omega_2^2 - \omega^2}\right) \tag{12}$$

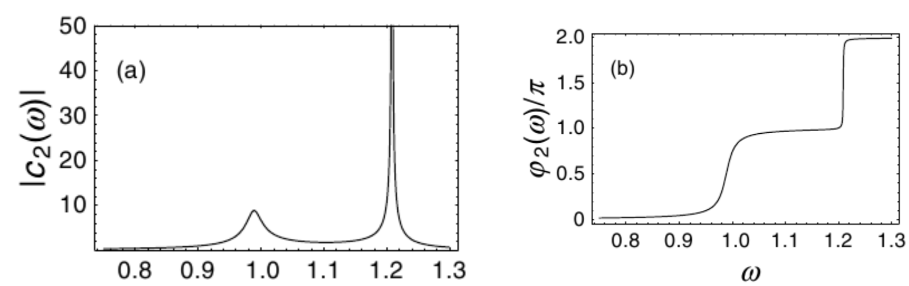
Now, we consider the case when **the damping parameter of the second oscillator is zero** i.e. $\gamma_2 = 0$, but $\gamma_1 \neq 0$. Just like the single oscillator system, the amplitudes of the oscillators are finite because of the presence of the effective damping. The below figures show the amplitude and phase of the first oscillator for $\gamma_1 = 0.025$ & $v_{12} = 0.1$:

Clearly the 1st oscillator has one **symmetric** resonance peak at $\omega \approx 1$ and another **asymmetric** peak at $\omega \approx 1.21$. The asymmetric peak near $\omega \approx 1.21$ occurs due to existence of the zero-frequency at $\omega_0 = \omega_2 \approx 1.2$, where the amplitude of the first oscillator becomes zero. This can be seen from (9). This causes the distortion of the amplitude of the first oscillator leading to the asymmetric peak.



Now we will try to understand the **physical reason** behind this asymmetry. Due to presence of the coupling between the two, the phases of the oscillators change as the driving frequency passes through the resonant frequency. When the driving frequency passes through the first resonant point, the amplitude of this oscillator grows quickly to maximum and the phase gets shifted by an angle $\pi/2$ (lags behind the driving force). Now before reaching the second resonance frequency namely the zero-frequency, the first oscillator settles down to steady state and eventually the phase of this oscillator becomes π i.e. out of phase with respect to the driving force. Next, as the external frequency passes through the anti-resonant at $\omega = \omega_0$, the phase of the first oscillator drops by π abruptly and again it jumps up around the resonance peak to phase π, thus gaining essentially no net phase shift after passing through the zero-frequency point.

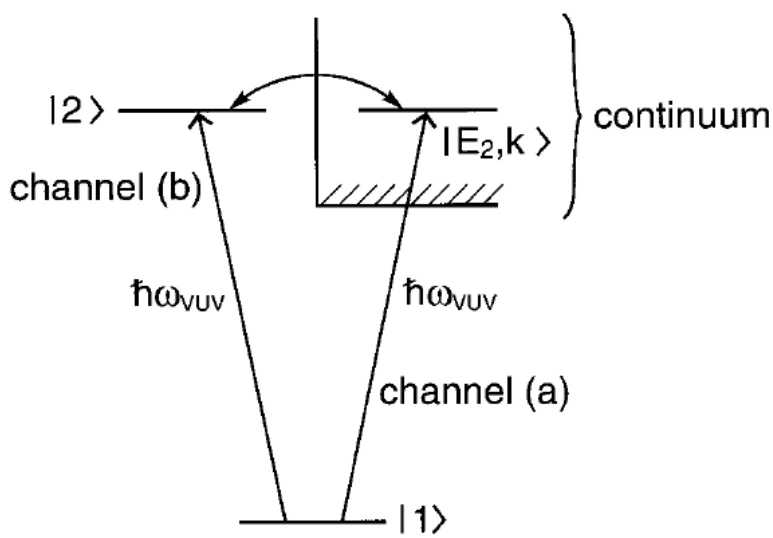
Similarly, if we examine the amplitude and the phase of the second oscillator we get the following curves:



Two symmetric resonance peaks appear and the phase gets shifted rather smoothly by π each time as the oscillator passes through the resonant point. So, as the driving frequency crosses the second resonant point, the phase of the second oscillator lags 2π behind the external force.

3 Fano Resonance in Electromagnetically Induced Transparency(EIT)

It has been known that if there is a possibility of making transition of some atom from one state to another via several alternative transition processes, interference between the probability amplitudes of these processes may lead to an enhancement(constructive interference) or a complete cancellation(destructive interference) of the total transition probability. The probability amplitudes, which are needed to be summed over to get the total transition probability, being either positive or negative can lead to such kind of phenomena. An example of this phenomenon in atomic systems is Fano interference, seen for radiative transitions to autoionizing states in atoms leading to an asymmetric line profiles.

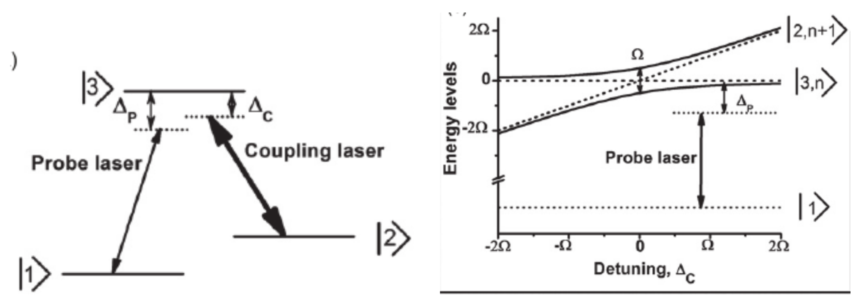


Here, the ground state $|1\rangle$ is coupled to the continuum state $|E_2, k\rangle$ via two alternative pathways: channel (a) is the direct photoionization path by absorption of an ultraviolet photon, whereas the channel (b) is the indirect photoionization path where the ultraviolet photon is absorbed to excite the atom into the bound state $|2\rangle$ (having the same energy as $|E_2, k\rangle$) followed by a transition to the continuum state via inter-electronic Coulomb interaction. The probability amplitudes of these two process must be summed over to get the overall transition probability amplitude. The interference would be either constructive or destructive depending on the frequency of the UV photon. If the frequency is such, that there is no absorption of the UV photons then this phenomenon is called EIT since the transparency is induced by the external electromagnetic field. This interference is also an example of Fano Interference.

4 Analogy between a Coupled Oscillator and Coherent Atom-Laser Interactions

There is an analogy between the classical oscillator system, which we have investigated in section (2) and a system of coherent atom-laser interaction.In order to see the connection between the asymmetric line-shape near the resonant frequency in the coupled oscillators and the main feature of the Fano interference associated in a quantum system, we study the Coherent Atom-Laser interaction.

We consider a three-level atomic system:

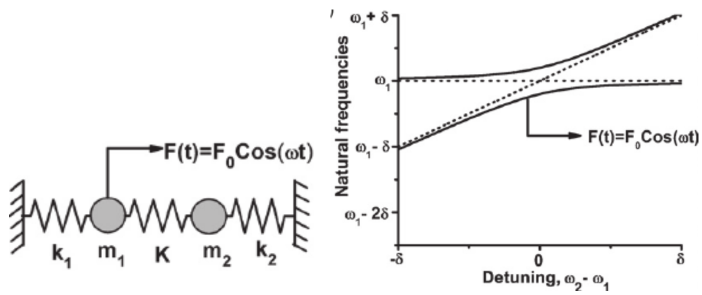


A strong coupling laser couples the transition $|2\rangle \rightarrow |3\rangle$. EIT as a result of Fano Interference takes place in the transition $|1\rangle \rightarrow |3\rangle$ because of the interference between the two alternative pathways : (1) probability amplitude of the transition from $|1\rangle \rightarrow |3\rangle$, and (2) probability amplitude of the transition from $|1\rangle \rightarrow |3\rangle \rightarrow |2\rangle \rightarrow |3\rangle$.

The probe and the coupling laser detuning(the difference between the laser frequency and the atomic resonant frequency) are $\triangle_P$ & $\triangle_C$ respectively. In the presence of the strong coupling laser, the bare states of the combined laser and the atom are $|3, n\rangle$ & $|2, n+1\rangle$, where n is the number of the photons present in the laser field while the atom is in the excited state $|3\rangle$. When the atom is de-excited to the state $|2\rangle$, then one photon is emitted resulting the total number of photons as $(n+1)$. These bare states cross each other at $\triangle_C = 0$ and becomes degenerate. But, due to the presence of the coupling the dressed states : $|+\rangle$(symmetric) & $|-\rangle$(anti-symmetric), which are linear superposition of the bare states, do not cross each other at

$\triangle_c = 0$ known as Autler-Townes splitting. EIT for the resonant probe laser is observed for $\triangle_c = 0$ because of the destructive interference of the channels $|1\rangle \to |+\rangle$ and $|1\rangle \to |-\rangle$. Resonance for the two-photon Raman transition is observed when the detuning of the coupling laser is larger than the natural line width of the $|1\rangle \to |3\rangle$ transition. In this case, the resonance displays an asymmetric line profiles.

Now, we will see exactly similar physics in the case of coupled harmonic oscillator system. In the figure given below, the natural frequencies of the oscillators are plotted as a function of detuning $\omega_2 - \omega_1$:

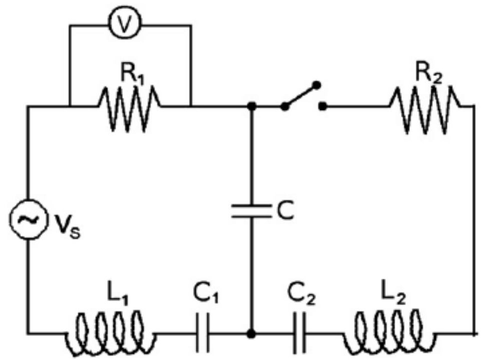


Two lines cross each other when $\triangle\ \omega = 0$ i.e. for zero tuning. But, as we have seen in the section (2.2), if we calculate the normal mode frequencies then those will avoid crossing at zero-detuning which is exactly similar to the dressed state picture for the coherent atom-laser interaction. The normal mode splitting at zero-detuning is analogous to the Autler-Townes splitting in the case of atom-laser interaction. Therefore, Fano resonance should be observed in case of damped coupled oscillator system upon applying sinusoidal driving force. EIT is observed for zero-detuning and the two-photo Raman transition is observed for non-zero detuning, $|\omega_2 - \omega_1| > 0$. Fano interference can be explained as the destructive interference of the two energy transfer channels from the source to the coupled oscillator through the normal-mode frequencies.

All important features of EIT can be observed when the normal-mode splitting $\omega_1^{(n)} - \omega_2^{(n)} >> \gamma_1, \gamma_2$, where γ_1, γ_2 are the damping constants of the oscillators respectively.

5 Fano Interference in a Coupled Electrical Resonator Circuit

As we know, the two systems: (1) driven damped coupled oscillator system, & (2) the electrical resonator circuit are physically equivalent in their behaviour. Hence, to demonstrate the Fano interference at the classical level we will use the coupled electrical resonator circuit.



The (L_1, C_1) & (L_2, C_2) represent the first & the second oscillator respectively. The C acts as the coupling spring between the two oscillators. The resistors R_1 & R_2 are analogous to the damping constant of the two oscillators respectively since they are responsible for the energy loss. The resonator circuit is driven by a sinusoidal voltage source.

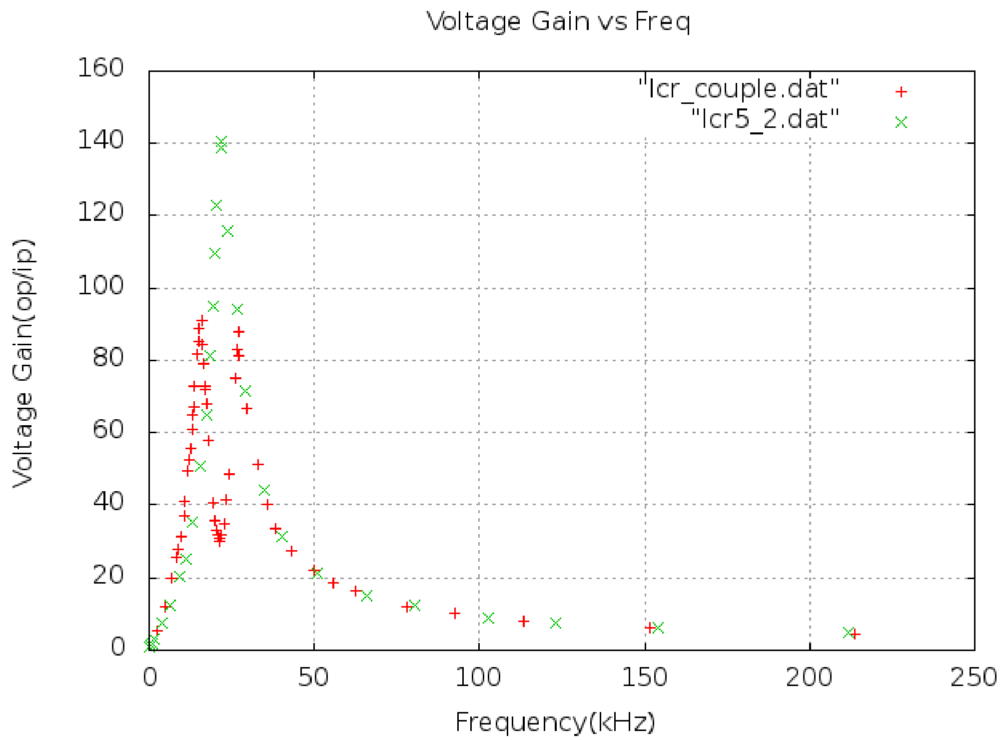
5.1 Experimental Demonstration

We experimentally demonstrate the EIT and the two-photon Raman transition in a coupled LCR resonator circuit.

- **Apparatus**
  $C_1 = 0.104\mu F, C_2 = 0.107\mu F, C = 0.125\mu F$
  $L_1 = 0.942mH, L_2 = 0.943mH$
  $R_2 = 0\Omega, R_1 = 5\Omega$
  Functional generator & CRO.

- **Experiment** 1 **: Voltage Gain vs Frequency**
  Firstly, we perform without the second part of the oscillator circuit and we get the Voltage gain across the resistor R_1 vs the Frequency of the external source plot which comes out to be asymmetric in nature

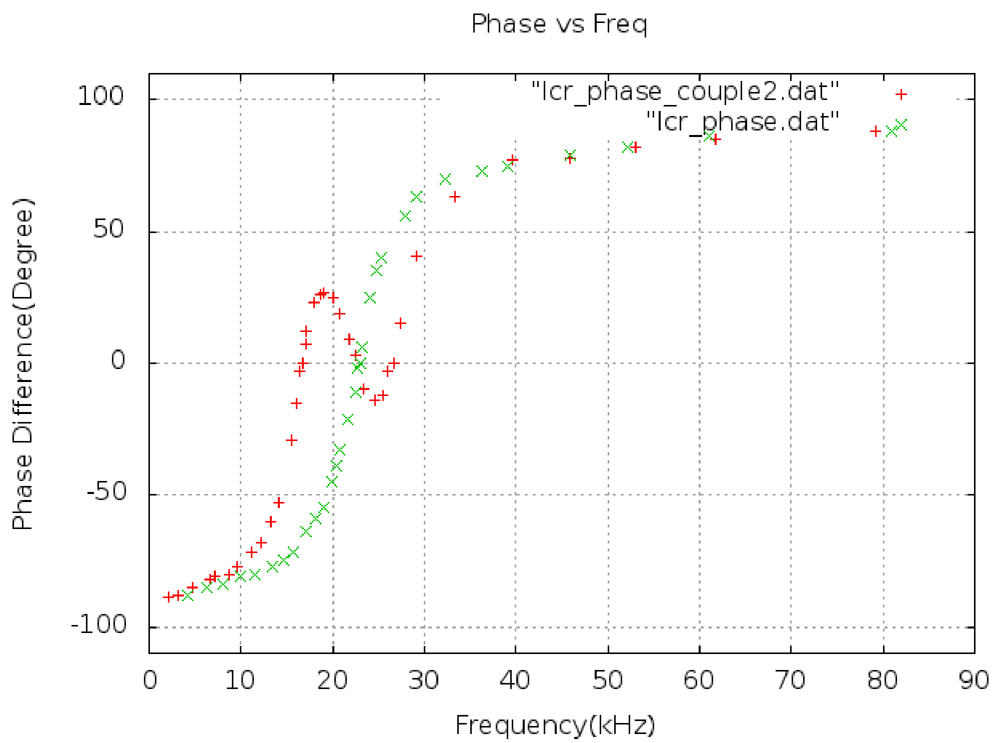
as we expect. Secondly, when we include the second part of the circuit with the coupling capacitor, then we get the EIT like behaviour.



The second resonance peak has typical typical Fano line shape, which is very much prominent since we make the resistance of the second part of the circuit zero. If the loss is included by inserting non-zero resistance R_2, then the Fano line shape gets washed out though it preserves the general feature of the resonance.
In the case of EIT, complete transparency is observed for $R_2 = 0$, even for very small coupling which leads to the normal-mode splitting being much smaller than the width of the resonance peak. This observation suggests that the observed transparency in this classical system is due to the Fano interference of the two channels(through which energy transfers to the system) and not due to the normal-mode doublet.

- **Experiment 2 : Phase Difference vs Frequency**
  We measure the relative phase with respect to the source voltage (phase of V_R − phse of V_S) with changing source frequency. The

graph, which is obtained, shows us the EIT like behaviour.



The EIT like behaviour is clearly evident in the resonance and the phase data. However, experimentally obtained Fano line shape is not that much of prominent because of the presence of spurious resistance in the circuit. Washing out of the Fano pattern with increasing R_2 gives us the confirmation of this argument.

- **Precautions & Discussions**
  Though the experiment is looking very simple but taking data correctly is not a trivial one. One should be careful enough regarding some issues, like as:

  1. To measure the voltage gain, theoretically one could have fixed the input peak to peak voltage of the functional generator and just measure the output voltage with varying frequency. But, one should not try to fix the input peak to peak voltage of the functional generator because as one increases the frequency, the

input voltage starts changing & near the resonance it changes so rapidly that one can't fix the input voltage to the desired value. Always one should take the ratio of the output-to-input voltage to get the voltage gain.

I think the primary reason for this phenomena is **skin effect**. Skin effect is the tendency of an alternating electric current (AC) to become distributed within a conductor such that the current density is largest near the surface of the conductor, and decreases with greater depths in the conductor. The electric current flows mainly at the "skin" of the conductor, between the outer surface and a level called the skin depth. The skin effect causes the effective resistance of the conductor to increase at higher frequencies where the skin depth is smaller, thus reducing the effective cross-section of the conductor. This decreases the quality factor of the circuit. The skin effect is due to opposing eddy currents induced by the changing magnetic field resulting from the alternating current.

2. One should not increase as well as decrease the frequency at a go to take the readings, that may not give the correct data. Rather one should take the data with increasing frequency till the last and then start decreasing the frequency & again take the data corresponding to the previous points. At last average should be taken.
3. Before start doing the experiment one should figure out the theoretical resonant frequency to make himself/herself highly careful near the resonance.
4. All the grounds of all the cables should be put to the same point in the bread-board.
5. In the beginning one should verify the connections of the bread-board.
6. Near the resonance, one should try to take the average reading judiciously because signal may be significantly distorted near the resonance.

6 Conclusion

We have discussed an analogy between the coupled classical oscillators and the dressed state picture of coherent atom-laser interactions.In the mechanical systems of the classical oscillator the Fano like behaviour can occur. Particularly, we have discussed how the asymmetry in the resonance curve arises. This analogy between the quantum system and the classical system helps us to understand the connection between the classical systems & the quantum phenomena like EIT.

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